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On the norm of the nilpotent residuals of all subgroups of a finite group ☆

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ABSTRACT

R. Baer and Wielandt in 1934 and 1958, respectively, considered the intersection of the normalizers of all subgroups of G and the intersection of the normalizers of all subnormal subgroups of G. In this paper, for a finite group G, we define the subgroup $S(G)$ to be the intersection of the normalizers of the nilpotent residuals of all subgroups of G. Set $S_0 = 1$. Define $S_{i+1}(G)/S_i(G) = S(G/S_i(G))$ for $i \geqslant 1$. By $S_\infty(G)$ denote the terminal term of the ascending series. It is proved that $G = S_\infty(G)$ if and only if the nilpotent residual $G^{\mathcal{N}}$ is nilpotent. Furthermore, if all elements of prime order of G are in $S(G)$, then G is solvable and $l_p(G) \leqslant 1$, where $l_p(G)$ is p-length of G for $p \in \pi(G)$, where $\pi(G)$ means the set of prime divisors of $|G|$.

1. Introduction

All groups considered in this paper are finite; the notation and terminology used in this paper are standard, as in [10,12]. $\mathcal{N}$ denotes the class of nilpotent groups. For the formation $\mathcal{N}$, each group has a smallest normal subgroup N such that G/N is in $\mathcal{N}$. This uniquely determined normal subgroup of G is called the nilpotent residual of G and is denoted by $G^{\mathcal{N}}$. In addition, $\mathcal{F}_{nn}$ denotes the class of finite groups G with $G^{\mathcal{N}}$ nilpotent, that is, G is a meta-nilpotent group. It is well known that $\mathcal{F}_{nn}$

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is a saturated formation containing all supersolvable groups. Moreover, H_G is the normal core of the subgroup H in G; $|G|$ denotes the order of G and $\pi(G)$ means the set of prime divisors of $|G|$.

By $N(G)$ denote the intersection of the normalizers of all subgroups of G and by $\omega(G)$ denote the intersection of the normalizers of all subnormal subgroups of G. Those concepts were introduced by R. Baer and H. Wielandt in 1934 and 1958, respectively, and were investigated by many authors, for example, see [1–5,7,9,16–18]. In [13], Li and Shen considered $D(G)$, where $D(G)$ is the intersection of the normalizers of the derived subgroups of all subgroups of G. That is

$$D(G) = \bigcap_{H \leqslant G} N_G\bigl(H'\bigr),$$

where H' is the derived subgroup of H.

In this paper, we consider the intersection of the normalizers of the nilpotent residuals of all subgroups of G and give the following:

Definition 1.1. Let G be a finite group. By $S(G)$ denote the intersection of the normalizers of the nilpotent residuals of all subgroups of G. That is

$$S(G) = \bigcap_{H \leqslant G} N_G\bigl(H^{\mathcal{N}}\bigr),$$

where $H^{\mathcal{N}}$ is the nilpotent residual of H.

Obviously, $S(G)$ is a characteristic subgroup of G. Let $Z_\infty(G)$ be the terminal term of the ascending central series of G. In the light of a theorem of P. Hall [12, III, Hauptsatz 2.11], $[G^{\mathcal{N}}, Z_\infty(G)] = 1$ and

$$Z_\infty(G) \leqslant C_G\bigl(G^{\mathcal{N}}\bigr) \leqslant S(G).$$

We give two examples on $S(G)$ as remarks, which are useful in Sections 3 and 4. Example 1.2 indicates that the subgroup $S(G)$ may be non-supersolvable. Example 1.3 shows that $D(G) < S(G)$ is possible when G is a nilpotent group.

Example 1.2. Assume that $G = [Q_8]Z_3 = SL(2,3)$, where Z_3 is a cyclic subgroup of $Aut(Q_8)$ of order 3. Then $S(G) = G$ and $G^{\mathcal{N}} = Q_8$ is non-abelian.

Proof. The non-nilpotent subgroup of G is G. Furthermore, $G^{\mathcal{N}} = Q_8$, which is normal in G. Hence $S(G) = G$, and $G^{\mathcal{N}}$ is non-abelian.

Example 1.3. As $Aut(Q_8) \cong S_4$ and $D_8 \leqslant S_4$, we have the semidirect product $G = [Q_8]D_8$. Then G is a 2-group of order 2^6 and $D(G) < G = S(G)$ since G is nilpotent.

The case that $S(G) = 1$ is possible for a solvable group G. For instance, the symmetry group S_4 of four letters satisfies $S(S_4) = 1$.

Theorem 1.4. *Let $G = A \times B$, where A and B are subgroups of G and $(|A|, |B|) = 1$. Then $S(G) = S(A) \times S(B)$.*

Proof. Let H be any subgroup of G. Since $(|A|, |B|) = 1$, we have $H = (H \cap A) \times (H \cap B)$. Therefore

$$H^{\mathcal{N}} = (H \cap A)^{\mathcal{N}} \times (H \cap B)^{\mathcal{N}}.$$

Hence

$$\begin{aligned} N_G\big(H^{\mathcal{N}}\big) &= N_G\big((H\cap A)^{\mathcal{N}}\big)\cap N_G\big((H\cap B)^{\mathcal{N}}\big)\\ &= \big(N_A\big((H\cap A)^{\mathcal{N}}\times B\big)\big)\cap\big(A\times N_B\big((H\cap B)^{\mathcal{N}}\big)\big)\\ &= N_A\big((H\cap A)^{\mathcal{N}}\big)\times N_B\big((H\cap B)^{\mathcal{N}}\big). \end{aligned}$$

Now the result follows. □

Notice the conclusion of Theorem 1.4 is false if one removes the additional condition, see the following example.

Example 1.5. Let $G = A \times B$ with $A \cong B \cong A_4$. We have $S(G) < S(A) \times S(B)$.

Proof. Let $A = \langle(123), (12)(34)\rangle$ and $B = \langle(567), (56)(78)\rangle$ Then $S(A) = A$ and $S(B) = B$. Consider the subgroup $L = \langle(123)(567), (12)(34)(56)(78)\rangle$. Then $L^{\mathcal{N}}$ is not normal in G, so $S(G) < S(A) \times S(B)$. □

Definition 1.6. For a finite group G, there exists a series of normal subgroups:

$$1 = S_0(G) \leqslant S_1(G) \leqslant S_2(G) \leqslant \cdots \leqslant S_n(G) \leqslant \cdots$$

satisfying $S_{i+1}(G)/S_i(G) = S(G/S_i(G))$ for $i = 0, 1, 2, \ldots$ and $S_n(G) = S_{n+1}(G)$ for some integer $n \geqslant 1$. Write $S_\infty(G)$ for the terminal term of the ascending series.

2. Basic properties

In this section we prove some basic properties of the subgroups $S(G)$ and $S_\infty(G)$.

Proposition 2.1. *If $M \leqslant G$, then $M \cap S(G) \leqslant S(M)$.*

Proof. Clearly, $S(G) = \bigcap_{H\leqslant G} N_G(H^{\mathcal{N}}) \leqslant \bigcap_{H\leqslant M} N_G(H^{\mathcal{N}})$. So

$$M \cap S(G) \leqslant M \bigcap_{H\leqslant M} N_G\big(H^{\mathcal{N}}\big) = \bigcap_{H\leqslant M} N_M\big(H^{\mathcal{N}}\big) = S(M). \qquad \square$$

Proposition 2.2. *If $N \trianglelefteqslant G$, then $S(G)N/N \leqslant S(G/N)$.*

Proof. Each subgroup of G/N possesses the form H/N, where H is a subgroup of G containing N and $(H/N)^{\mathcal{N}} = H^{\mathcal{N}}N/N$. For any element $x \in S(G)$, by definition, x normalizes $H^{\mathcal{N}}$. It follows that xN normalizes $H^{\mathcal{N}}N/N$. Thus every element of $S(G)N/N$ normalizes $(H/N)^{\mathcal{N}}$ for all subgroups H/N of G/N, so $S(G)N/N \leqslant S(G/N)$. □

Proposition 2.3. *Let $N \trianglelefteqslant G$ and $N \leqslant S_\infty(G)$. Then $S_\infty(G/N) = S_\infty(G)/N$.*

Proof. As $N \leqslant S_\infty(G)$, $N \leqslant S_i(G)$ for some i. Set $S^1(G)/N = S(G/N)$ and by $S^\infty(G)/N$ denote the terminal term of the ascending series of G/N. We claim that $S^1(G) \leqslant S_{i+1}(G)$. For any subgroup $H/S_i(G)$ of $G/S_i(G)$, H/N is a subgroup of G/N. By definition, any element $x \in S^1(G)$ normalizes $(H/N)^{\mathcal{N}} = H^{\mathcal{N}}N/N$, namely $(H^{\mathcal{N}})^x N/N = H^{\mathcal{N}}N/N$. As $N \leqslant S_i(G)$, of course, we have $(H^{\mathcal{N}})^x S_i(G)/S_i(G) = H^{\mathcal{N}} S_i(G)/S_i(G)$, so x normalizes $(H/S_i(G))^{\mathcal{N}}$. Therefore $x \in S_{i+1}(G)$. The claim holds. Now, by induction, we gave $S^\infty(G) \leqslant S_\infty(G)$. Conversely, clearly $S(G) \leqslant S^1(G)$, by induction we have $S_\infty(G) \leqslant S^\infty(G)$. Consequently, $S_\infty(G/N) = S_\infty(G)/N$, completing the proof. □

Proposition 2.4. *For any group X, the subgroup $S_\infty(X)$ is solvable.*

Proof. Clearly, we need only to show that $S(X)$ is solvable. Write $G = S(X)$. Then G has the property: the nilpotent residual of every subgroup of G is normal in G. Let M be a maximal subgroup of G. If $M^{\mathcal{N}} > 1$, then $M^{\mathcal{N}} \trianglelefteq G$. By Propositions 2.1, 2.2 and the induction, $G/M^{\mathcal{N}}$ and $M^{\mathcal{N}}$ are solvable, hence G is solvable. Thus every maximal subgroup of G is nilpotent. By Schmidt–Iwasawa's theorem, G is solvable. □

Theorem 2.5. *Let G be a group. Then $G^{\mathcal{N}} \cap S(G) = 1$ if and only if $Z(G^{\mathcal{N}}) = 1$.*

Proof. $\Rightarrow$: As $C_G(G^{\mathcal{N}}) \leqslant S(G)$ holds, the conclusion follows.

$\Leftarrow$: Assume that $G^{\mathcal{N}} \cap S(G) > 1$. By Proposition 2.4, $S(G)$ is solvable, so we can find a minimal normal subgroup N of G such that $N \leqslant G^{\mathcal{N}} \cap S(G)$. Then N is an elementary abelian p-subgroup for some prime p. Let P be a Sylow p-subgroup of G, set $C = C_G(N)$ and consider the factor group

$$\overline{G} = G/C.$$

We claim the following conclusions.

Step 1. For any prime q with $q \neq p$, G/C is q-nilpotent.

Suppose contrary for some fixed q, so that G/C is a non-q-nilpotent. Thus there exists a non-q-nilpotent subgroup K/C all of whose proper subgroups are q-nilpotent. Choose a subgroup L of K such that $K = CL$ and $C \cap L \leqslant \Phi(L)$. Then $L/\Phi(L) \cong K/C$, so $L/\Phi(L)$ is a minimal non-q-nilpotent group. In the light of a theorem of Itô [14, p. 296, Theorem 10.3.3], $L/\Phi(L) = Q/\Phi(L) \cdot R/\Phi(L)$ is a minimal non-nilpotent group of order $q^m r^n$, where $Q/\Phi(L)$ is a normal Sylow q-subgroup of $L/\Phi(L)$ and $R/\Phi(L)$ is a cyclic Sylow r-subgroup of $L/\Phi(L)$, $r \neq q$ is a prime, $m, n \geqslant 1$.

Now $(L/\Phi(L))^{\mathcal{N}}$ is an q-group, so $L^{\mathcal{N}}$ is nilpotent. Let Q_0 be a Sylow q-subgroup of $L^{\mathcal{N}}$. Then we have Q_0 char $L^{\mathcal{N}}$. By definition of $S(G)$, N normalizes $L^{\mathcal{N}}$, so normalizes Q_0. As Q_0 is a q-subgroup of G with $q \neq p$, we know that $[N, Q_0] = 1$, and so $Q_0 \leqslant C$. Consequently, $Q_0 \leqslant C \cap L \leqslant \Phi(L)$ and hence $L/\Phi(L)$ is nilpotent, a contradiction.

Step 2. G/C is p-closed, that is, the Sylow p-subgroup CP/C of G/C is normal.

By step 1, G/C is q-nilpotent, so let $H(q)/C$ be the normal Hall q'-subgroup of G/C for every prime q with $q \neq p$. Write

$$T/C = \bigcap_{q \neq p} H(q)/C.$$

As $H(q)/C$ is a q'-group and $P \leqslant H(q)$ for all q, we see that T/C is a Sylow p-subgroup of G/C and normal, as desired.

Step 3. $G^{\mathcal{N}} \leqslant T$.

By step 2, G/T is a p'-group. Then there exists a p'-subgroup W of G such that $G = TW$. Then N normalizes $W^{\mathcal{N}}$ and hence centralizes $W^{\mathcal{N}}$. Consequently, $W^{\mathcal{N}} \leqslant C$, and it follows that G/T is nilpotent. Thus $G^{\mathcal{N}} \leqslant T$.

Step 4. Finishing the proof.

As P is a Sylow p-subgroup of T containing N, then $N \cap Z(P) \neq 1$, there exists an element x such that $x \in N \cap Z(P)$ and $x \neq 1$. Now $C \leqslant C_G(x)$ and $P \leqslant C_G(x)$, it follows that $T \leqslant C_G(x)$. Consequently, $G^{\mathcal{N}} \leqslant T \leqslant C_G(x)$ and we have $x \in Z(G^{\mathcal{N}})$, contrary to $Z(G^{\mathcal{N}}) = 1$. □

3. $\mathcal{F}_{nn}$-groups

In the section, groups in $\mathcal{F}_{nn}$ are characterized by means of the subgroup $S_\infty(G)$.

Lemma 3.1. *The following properties of the group G are equivalent*:

(i) $G \in \mathcal{F}_{nn}$;
(ii) $G/\Phi(G) \in \mathcal{F}_{nn}$.

We can now give a new characterization of $\mathcal{F}_{nn}$-groups.

Theorem 3.2. *Let G be a finite group. Then the following statements are equivalent*:

(i) $G \in \mathcal{F}_{nn}$;
(ii) $G/S(G) \in \mathcal{F}_{nn}$.

Proof. (i) $\Rightarrow$ (ii): Clear.

(ii) $\Rightarrow$ (i): We use induction on the order of G. If $S(G) = 1$, then nothing needs to be shown. Suppose that $S(G) > 1$. So that we can find a minimal normal subgroup N of G such that $N \leqslant S(G)$. By Proposition 2.4, $S(G)$ is solvable, so N is an elementary abelian p-group for some prime p.

Firstly let $N \leqslant \Phi(G)$, the Frattini subgroup of G. By Lemma 2.2, $S(G)/N \leqslant S(G/N)$. It follows that $(G/N)/S(G/N)$ is in $\mathcal{F}_{nn}$ since $G/S(G) \in \mathcal{F}_{nn}$. We thus have that G/N satisfies the condition of the theorem. By induction, $(G/N)^{\mathcal{N}} = G^{\mathcal{N}}N/N$ is nilpotent. As $N \leqslant \Phi(G)$, it follows by [12, III, Satz 3.5] that $G^{\mathcal{N}}N$ is nilpotent and hence $G^{\mathcal{N}}$, which gives $G \in \mathcal{F}_{nn}$, as desired.

Nextly let $N \nsubseteq \Phi(G)$. Then there is a maximal subgroup M of G such that $G = NM$ with $N \cap M = 1$. By Proposition 2.1, $M \cap S(G) \leqslant S(M)$. Thus, by hypothesis that $G/S(G) \in \mathcal{F}_{nn}$, and as $G/S(G) \cong M/(S(G) \cap M)$, we have $M/S(M) \in \mathcal{F}_{nn}$. Hence M satisfies the condition. By induction, $M^{\mathcal{N}}$ is nilpotent. Now, as $N \leqslant S(G)$ and $S(G)$ normalizes the nilpotent residuals of all subgroups of G. Thus $M^{\mathcal{N}}$ is normal in G and it follows that $NM^{\mathcal{N}} = N \times M^{\mathcal{N}}$. Since $M^{\mathcal{N}}$ is nilpotent, we conclude that $G^{\mathcal{N}}$ is nilpotent, as desired. □

Theorem 3.3. *Let G be a finite group. Then the following statements are equivalent*:

(i) $G \in \mathcal{F}_{nn}$;
(ii) $(G/S_\infty(G)) \in \mathcal{F}_{nn}$;
(iii) $G = S_\infty(G)$;
(iv) $S(G/N) > 1$ *for any proper normal subgroup N of G.*

Proof. (i) $\Rightarrow$ (ii): Clear.

(ii) $\Rightarrow$ (iii): We first observe the following simple fact: If $X > 1$ is an $\mathcal{F}_{nn}$-group, then $S(X) > 1$. In fact, $X^{\mathcal{N}}$ is nilpotent, so $C_X(X^{\mathcal{N}}) > 1$. But $C_X(X^{\mathcal{N}}) \leqslant S(X)$, we have $S(X) > 1$. Using this fact and noting that $S(G/S_\infty(G)) = 1$, we deduce $G = S_\infty(G)$.

(iii) $\Rightarrow$ (i): As $S_\infty(G/S(G)) = S_\infty(G)/S(G)$, by induction, $G/S(G) \in \mathcal{F}_{nn}$. It follows by Theorem 3.2 that $G \in \mathcal{F}_{nn}$.

(i) $\Rightarrow$ (iv): See the argument of (ii).

(iv) $\Rightarrow$ (iii): By definition, $S(G/S_i(G)) = S_{i+1}(G)/S_i(G)$. As $S(G/S_i(G)) > 1$ by hypothesis, i.e., $S_{i+1}(G) > S_i(G)$ for $i = 0, 1, 2, \ldots$. So the terminal term $S_\infty(G)$ of the ascending series must be G. □

By [6, Corollary 6], we have:

Theorem 3.4. *If H is an $\mathcal{F}_{nn}$-subgroup of the finite group G, then $S_\infty(G)H$ is an $\mathcal{F}_{nn}$-group. Consequently, $S_\infty(G)$ is contained in every maximal $\mathcal{F}_{nn}$-subgroup of G.*

4. S-groups

Definition 4.1. A group G is called an S-group if $G = S(G)$, that is, the nilpotent residuals of all subgroups of G are normal.

The following facts are clear from Definition 4.1:

Proposition 4.2.

(i) *The subgroups of an S-group are S-groups*;
(ii) *The quotient groups of an S-group are S-groups*;
(iii) *If G is a nilpotent group or a minimal non-nilpotent group, then G is an S-group.*

Lemma 4.3. *(See Robinson [15] and Hall [11].) Let G be a group. Then the following statements are true*:

(i) *G is a supersolvable group if and only if there is a normal nilpotent subgroup N of G such that G/N' is a supersolvable group.*
(ii) *G is a nilpotent group if and only if there is a normal nilpotent subgroup N of G such that G/N' is a nilpotent group.*

By Lemma 4.3, we have:

Theorem 4.4. *Let G be a group. Then the following statements are true*:

(i) *Assume G is an S-group. Then G is a supersolvable group if and only if $G/(G^{\mathcal{N}})'$ is a supersolvable group.*
(ii) *G is a nilpotent group if and only if G is an S-group and $G/(G^{\mathcal{N}})'$ is a nilpotent group.*

Theorem 4.5. *If $G/(G^{\mathcal{N}})'$ is a minimal non-nilpotent group and $G^{\mathcal{N}}$ is of nilpotency class 2, then G is an S-group.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. By hypothesis, there is $M < G$ such that $M^{\mathcal{N}}$ is not normal in G. Consider $N_G(M^{\mathcal{N}})$. If $G^{\mathcal{N}}$ is of nilpotency class 2, then $(G^{\mathcal{N}})' \leqslant Z(G^{\mathcal{N}})$. Since $M^{\mathcal{N}} \leqslant G^{\mathcal{N}}$, $G > N_G(M^{\mathcal{N}}) \geqslant M^{\mathcal{N}}(G^{\mathcal{N}})'$. Therefore, $M(G^{\mathcal{N}})'/(G^{\mathcal{N}})'$ is nilpotent as $G/(G^{\mathcal{N}})'$ is a minimal non-nilpotent group. By Lemma 4.3, $M(G^{\mathcal{N}})'$ is nilpotent and so M. This is a contradiction. □

Theorem 4.6. *If $G/(G^{\mathcal{N}})_3$ is a minimal non-nilpotent group and $G^{\mathcal{N}}$ is of nilpotency class 3, then G is an S-group.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. By hypothesis, there is $M < G$ such that $M^{\mathcal{N}}$ is not normal in G. Consider $N_G(M^{\mathcal{N}})$. If $G^{\mathcal{N}}$ is of nilpotency class 3, then $(G^{\mathcal{N}})_3 \leqslant Z(G^{\mathcal{N}})$. Since $M^{\mathcal{N}} \leqslant G^{\mathcal{N}}$, $G > N_G(M^{\mathcal{N}}) \geqslant M^{\mathcal{N}}(G^{\mathcal{N}})_3$. Therefore, $M(G^{\mathcal{N}})_3/(G^{\mathcal{N}})_3$ is nilpotent as $G/(G^{\mathcal{N}})_3$ is a minimal non-nilpotent group. By Lemma 4.3, $M(G^{\mathcal{N}})_3$ is nilpotent and so M. This is a contradiction. □

Theorem 4.7. *If G is a supersolvable S-group, then the nilpotent residual $G^{\mathcal{N}}$ is abelian.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. Since G is supersolvable, by hypothesis, G has unique minimal normal subgroup N of order p, where p is the largest prime divisor of $|G|$. By the induction hypothesis and Proposition 2.2, $G^{\mathcal{N}}/N$ is abelian and $O_{p'}(G) = 1$. Let P be Sylow p-subgroup of G. Then $G = [P]H$, where H is a p'-Hall subgroup.

Since $G/[P,H]$ is nilpotent, $G^{\mathcal{N}}=[P,H]$. Moreover, $[P,H,H]=[P,H]$ and the minimality of G imply $G^{\mathcal{N}}=P$. Hence $G^{\mathcal{N}}=F(G)$.

Assume $G^{\mathcal{N}}$ is not cyclic, then there is the normal subgroup U of G of type $Z_p\times Z_p$ and $U=N\times V$. We consider UH. Since G is supersolvable, V is also H-invariant. If H acts nontrivially on V, then V is the nilpotent resdual of VH. By hypothesis, V is normal in G, which contradicts to the uniqueness of N. Therefore, H acts trivial on V and so H acts trivially on U/N. Hence $C_{G^{\mathcal{N}}/N}(H)>1$. This is a contradiction as the nilpotent resdual of G/N is $G^{\mathcal{N}}/N$. □

5. The dual of the PN-groups

Gaschütz and Itô proved that if all minimal subgroups of a group G are normal (which are called PN-groups), then G is solvable and the Fitting length of G is at most 3 ([12, p. 436, Satz 5.7] or [8]). In [13], Li and Shen considered the following problem: study the finite groups all of whose minimal subgroups normalize each derived subgroup of every subgroup of G.

In this section, we study the finite groups all of whose minimal subgroups normalize the nilpotent residual of every subgroup of G.

Remark. The idea of the result of Theorem 5.1 is attributed to Beidleman and Heineken [6, Lemma 10].

Theorem 5.1. *Let G be a p-solvable group. Suppose that all elements of G of order p are in $S(G)$. If $p=2$, in addition, all elements of G of order 4 are in $S(G)$. Then the $l_p(G)\leqslant 1$.*

Proof. We use induction on $|G|$. Clearly, $G/O_{p'}(G)$ satisfies the hypothesis and $l_p(G/O_{p'}(G))=l_p(G)$. We may assume that $O_{p'}(G)=1$.

Let P be a Sylow p-subgroup of $S(G)$. By Theorem 3.2, $S(G)^{\mathcal{N}}$ is nilpotent. Thus $O_{p'}(G)=1$ implies $S(G)^{\mathcal{N}}$ is a p-group, and hence P is normal in G. Also, $F_p(G)=O_{p',p}(G)=O_p(G)$. As G is p-solvable by the condition, by [14, p. 269, Theorem 9.3.1], we know

$$C_G\big(O_p(G)\big)\leqslant O_p(G).$$

We now claim that G is q-nilpotent for any prime $q\neq p$. Otherwise, there exists a prime q such that G is non-q-nilpotent. Then there exists a subgroup K with the following properties: K is non-q-nilpotent but all proper subgroups of K are q-nilpotent. By a theorem of Itô [14, p. 296, Theorem 10.3.3], $K=[Q]R$, where Q is a normal q-subgroup, $exp(Q)=p$ or 4, and R is a cyclic r-subgroup, the prime $r\neq q$. We know that $K^{\mathcal{N}}=Q$. Consider the subgroup

$$M=O_p(G)Q.$$

Let $p>2$. By above, $\Omega_1(G_p)\leqslant P\leqslant O_p(G)$, so $\Omega_1(G_p)=\Omega_1(O_p(G))$. Then $\Omega_1(O_p(G))\trianglelefteq G$. By hypothesis, $\Omega_1(O_p(G))$ normalizes $K^{\mathcal{N}}=Q$, it follows that $[Q,\Omega_1(O_p(G))]=1$. By [12, p. 437, 5.12], we get $[Q,O_p(G)]=1$. Thus $Q\leqslant C_G(O_p(G))$. As $C_G(O_p(G))\leqslant O_p(G)$ and Q is a p'-group, Q must be 1, a contradiction. Similar for the case when $p=2$.

Now let $G_{q'}$ denote the normal q-complement of G for every prime $q\neq p$. Then $G_p\leqslant G_{q'}$ and G_p is the intersection of all $G_{q'}$, hence $G_p\trianglelefteq G$, of course, $l_p(G)=1$. The proof is now complete. □

Theorem 5.2. *Let G be a finite group. If all elements of prime order of G are in $S(G)$, then*:

(i) *G is solvable*;
(ii) *The p-length of G is at most* 1 *for every odd prime p*; *and*
(iii) *The fitting length of G is bounded by* 3.

Proof. We firstly show (i). Assume that the theorem is false and let G be a counterexample of minimal order. If M is a proper subgroup of G, by Lemma 2.2 we have $M \cap S(G) \leqslant S(M)$. Thus all cyclic subgroups of M of odd prime order are in $S(M)$. So M satisfies the condition. By the choice of G, M is solvable. Consequently, G is a non-solvable group in which all proper subgroups are solvable, so that $G/\Phi(G)$ is a minimal simple group. As $S(G)$ is normal in G and solvable, it follows that $S(G) \leqslant \Phi(G)$, the Frattini subgroup of G.

By a theorem of Baer [10, Theorem 3.8.2], there exist two elements $x, y \in G$ with the following properties: $x^2, y^2 \in \Phi(G)$, $[x, y] \notin \Phi(G)$ but $[x, y]^k \in \Phi(G)$ for some odd integer. In particular, we may assume that k is a prime. Consider the proper subgroup $\langle x, y\rangle$ (in fact, $\langle x\Phi(G), y\Phi(G)\rangle/\Phi(G)$ is a dihedral subgroup of $G/\Phi(G)$), this subgroup is solvable and possesses a Hall-$\{2, k\}$ subgroup, we call it H. Let K be a minimal supplement of $H \cap \Phi(G)$ in H. This is a supersolvable subgroup of G, and $H^{\mathcal{N}}$ is a cyclic k-subgroup. Choose an odd prime q different from k which divides $|G : \Phi(G)|$, such a prime exists since the order of a non-abelian simple group is divisible by at least three primes. Let Q be the Sylow q-subgroup of $\Phi(G)$, we know that $\Omega_1(Q) = \Omega_1(G) \cap Q$. Now $\Omega_1(Q)$ normalizes $H^{\mathcal{N}}$ by definition, also $H^{\mathcal{N}}\Omega_1(Q) \cap Q = \Omega_1(Q)$ is a normal subgroup of $H^{\mathcal{N}}\Omega_1(Q)$. This shows that $\Omega_1(Q)$ centralizes $H^{\mathcal{N}}$ and thus all conjugates of $H^{\mathcal{N}}$, but $(H^{\mathcal{N}})^G = G$. Now $\Omega_1(Q) \leqslant Z(G)$ and all elements of order q are contained in $Z(G)$ by hypothesis. Itô's lemma [12, p. 435, Satz 5.5] shows that G is q-nilpotent and not perfect, a contradiction. The proof of the solvability of G is now complete.

Now, conclusion (ii) follows from conclusion (i) and Theorem 4.1.

The proof of (iii): Let p be any odd prime dividing $|G|$ and let P be a Sylow p-subgroup of G. As G is solvable, hence p-solvable. According to (ii), we have $F_p(G) = O_{p',p}(G) = O_{p'}(G)P$, the maximal normal p-nilpotent subgroup of G. Then $C_G(P) \leqslant F_p(G)$ by [14, p. 269, Theorem 9.3.1]. Nextly, by Frattini argument $G = N_G(P)O_{p'}(G)$. On the other hand, by Schur–Zassensaus's theorem [15, p. 253, Theorem 9.1.2], $N_G(P) = [P]M$, where M is a Hall p'-subgroup of $N_G(P)$. By hypothesis, $\Omega_1(P)$ normalizes $M^{\mathcal{N}}$. Hence $M^{\mathcal{N}}$ centralizes $\Omega_1(P)$, and thus centralizes P. Consequently

$$M^{\mathcal{N}} \leqslant F_p(G).$$

Now $G = F_p(G)M$, it follows that $G/F_p(G) \cong M/F_p(G) \cap M$. This is a nilpotent group. Set T for the intersection of all $F_p(G)$. Then T is p-nilpotent for every odd prime p, and hence T is an extension of a 2-group by a nilpotent group of odd order. Thus we get a series of normal subgroups of G:

$$1 \leqslant T_2 \leqslant T \leqslant G,$$

where T_2 is the Sylow 2-subgroup of T. In this series all the factor groups are nilpotent, which indicates the Fitting length of G is at most 3, completing the proof. □

Theorem 5.3. *Let G be a finite group. If all elements of G of order prime or* 4 *are in $S(G)$, then*:

(i) *G is solvable*;
(ii) *$l_p(G) \leqslant 1$ for every prime p*; *and*
(iii) *the Fitting length of G is bounded by* 3.

Proof. This follows from Theorem 5.1 and Theorem 5.2. □

Let us compare Theorem 5.3 with the following well-known result: If all the cyclic subgroups of a group G of prime order or order 4 are normal, then G is supersolvable. The previous Example 1.2 shows that the supersolvable conclusion cannot be expected under the condition of Theorem 5.3. □

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